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Matthew Gilbert, Linwei He and Thomas Pritchard. 2015.
The Structural Engineer, 93 (10), 44-48.

The yield-line method for concrete slabs: automated at last.

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Synopsis

The yield-line method of analysis provides a powerful means of identifying the ultimate load carrying capacity of reinforced concrete slabs. Benefits of the yield-line method are that it will often identify additional reserves of strength when applied to the analysis of existing slabs, and to highly economic slabs when used in design. Traditionally a hand-based method, the yield-line method is easy to apply to problems involving simple slab geometries and loading regimes. However, when these become more complex it can be difficult to identify the critical yield-line pattern. To address this, the method has now been systematically automated. The automated method quickly identifies the critical mechanism (or a close approximation of this) and corresponding load carrying capacity, providing engineers with a powerful new computer-based tool for the analysis and design of concrete slabs. In this article the discontinuity layout optimisation (DLO) procedure which has been used to automate the yield-line method is briefly described and then applied to various example problems.

Introduction

Reinforced concrete slabs are a feature of many modern buildings and bridges. When designing or assessing a reinforced concrete slab, elastic analysis methods have become popular in recent years, largely due to the availability of efficient computer-based implementations (e.g. using grillage analysis or finite element analysis techniques). Elastic methods are therefore now often used both to estimate slab deflections under service loads (to establish the serviceability limit state, SLS) and to analyse a slab at failure (to establish the ultimate limit state, ULS).

However, a standard elastic analysis does not take account of the redistribution of moments that takes place after yielding of the reinforcement in a slab. This means that an elastic analysis may provide a grossly over-conservative estimate of ULS capacity. In cases when the ULS is critical this is likely to lead to more material (i.e. more concrete and/or steel reinforcement) being specified in a design than is necessary. To address this a non-linear analysis (e.g. a non-linear finite element analysis) could be performed; however this type of analysis tends to be demanding in terms of operator expertise and computer resources, and is generally not considered suitable for routine use. Alternatively a much simpler plastic analysis method, such as the yield-line method, could be used. However, the lack of an efficient computer-based implementation of the yield-line method has reduced its popularity in recent years.

The term 'yield-line' was first coined by Ingerslev¹ in the very first article to appear in *The*

Structural Engineer in 1923. Subsequently Johansen² developed the theory underpinning the general yield-line method, later shown to be an ‘upper bound’ plastic analysis method within the context of the then emerging plastic theorems³. Since a reinforced concrete slab generally contains a low percentage of reinforcement, the section will generally yield in flexure in a ductile manner, thereby justifying the use of plastic methods. Benefits of the yield-line method are that it will often identify additional reserves of strength when applied to the analysis of existing slabs, and to highly economic slabs when used in design⁴.

The traditional hand-based method involves postulating a yield-line pattern (failure mechanism) and then using the work method to compute the corresponding load carrying capacity. However, due to the upper-bound nature of the yield-line method, a range of yield-line patterns will often need to be explored, which can be time-consuming. Furthermore, there is often the concern that the critical pattern may have been missed, and consequently that an unsafe estimate of load carrying capacity has been computed. This has prompted many practitioners to turn to computer-based elastic methods, which provide demonstrably safe, albeit frequently over-conservative, ULS predictions.

However, the yield-line analysis method has recently been systematically automated, thereby allowing the critical yield-line pattern (or a very close approximation of this) to be reliably found. This means that practitioners can now apply the yield-line method with confidence, even when slabs with complex geometries and/or loading regimes are involved. Since the yield-line method considers only flexural failure, additional checks (e.g. for punching shear failure and/or serviceability limit state deflections) will however still be required.

This article briefly describes the traditional process of yield-line analysis and then explains how it has been possible to systematically automate this. The new automated method is then applied to various practical example problems.

Traditional hand-based yield-line analysis

The yield-line method can straightforwardly be applied to problems involving simple slab geometries and loading regimes. The first step is to postulate a yield-line pattern, following basic rules to ensure that this is geometrically compatible (e.g. see Kennedy and Goodchild⁴). Figure 1 shows a sample yield-line pattern for a reinforced rectangular slab with two simple supports and two free edges, and subject to uniform pressure loading.

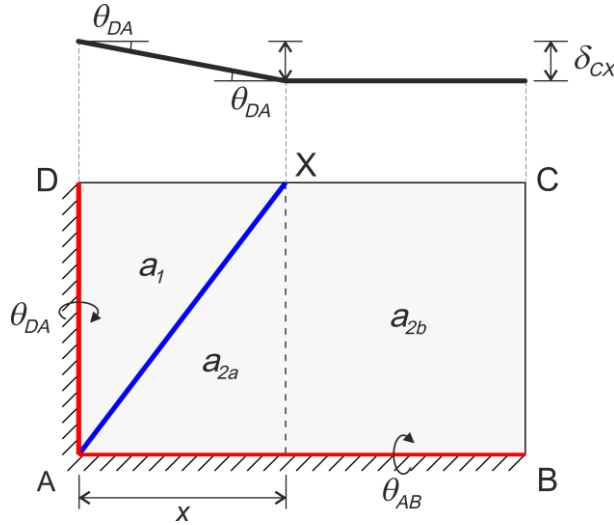


Figure 1: Example yield-line pattern for a rectangular slab with simple supports and free edges

The second step involves performing calculations to determine the load or load factor required to cause collapse (or, in the case of design, the moment capacity required to support the applied load). The work method is commonly used to do this; in this case the mechanism is perturbed and external work done by applied loads is equated to the internal work done along yield-lines:

External work (E) = Internal work (dissipation, D)

$$\lambda \times \sum_{\text{all slab regions}} q \times a \times \delta(\theta) = \sum_{\text{all yield-lines}} m_p \times l \times |\theta| \quad (1)$$

Where λ is a load factor, to be determined, q is the specified pressure loading per unit area, a is the area of a given rigid slab region, $\delta(\theta)$ is the displacement of the centroid of this slab region, which can be expressed as a function of yield-line rotations θ . Also m_p is the plastic moment of resistance per unit length of the slab, and l is the length of a given yield-line.

For the example shown in Figure 1, assuming lengths $AB = 9\text{m}$, $BC = 6\text{m}$, and isotropic moment of resistance m_p , sample calculations are given below.

External work:

$$E = \lambda q (a_1 \delta_1 + a_{2a} \delta_{2a} + a_{2b} \delta_{2b}) = \lambda q \theta_{AB} l_{BC} (a_1/3 + a_{2a}/3 + a_{2b}/2) = \lambda q \theta_{AB} (162 - 6x) \quad (2)$$

Internal work:

$$D = m_p l_{DA} \theta_{DA} + m_p x \theta_{AB} = m_p \theta_{AB} (36/x + x) \quad (3)$$

To determine the minimum load factor λ , the critical value of x must be found. For small problems this can be done by calculus or by trial and error (e.g. for this example x can be found to be 4.813m, and, when $m_p = 20\text{kNm/m}$ and $q = 1\text{kN/m}^2$, the computed load factor can be calculated to be 1.847). For larger problems, mathematical optimisation can be used; in this

case it is usual to pose the problem in a slightly different way, setting the unfactored external work done by the external loads to unity, giving the following mathematical optimisation problem:

$$\begin{aligned} & \text{minimise} \quad \lambda = \sum m_p l (\theta^+ + \theta^-) \\ & \text{subject to} \quad \sum qa\delta(\theta) = 1 \end{aligned} \tag{4}$$

Equation (4) is equivalent to Eq. (1), but with the additional stipulation that the value of λ is to be minimised. Also, to ensure positive dissipation along yield-lines, new variables θ^+ , θ^- have been introduced, where $\theta = \theta^+ - \theta^-$, and where θ^+ , $\theta^- > 0$. This modified form will be used in the next section.

The new automated method

In the simple yield-line analysis problem considered in the previous section a geometrically compatible yield-line pattern was pre-defined, and all that was required was to adjust the geometry of the mechanism (i.e. the distance x) to determine the critical case. However, in general, the critical yield-line pattern will not be known in advance, and the challenge is to identify this (or a good approximation of this) from a large set of possible geometrically compatible patterns.

One possible approach is to discretise the slab using rigid finite elements, with potential yield-lines lying along element boundaries^{5,6}. However, with this approach the set of geometrically compatible patterns from which the most critical can be chosen will be relatively small. Also, specially tailored meshes must be used in order to e.g. identify fan type mechanisms, which is clearly unsatisfactory.

An alternative approach involves considering the yield-line discontinuities directly, and enforcing the geometric compatibility requirement at the end points of yield-lines (nodes). Denton⁷ demonstrated that compatibility can be enforced for a yield-line mechanism in essentially the same way as equilibrium is enforced in a truss (Figure 2). Gilbert et al.⁸ then showed that the long-established 'layout optimisation' technique, used to identify the optimum topology of a truss, could also be used to identify the critical yield-line pattern, and corresponding collapse load (or load factor) of a slab. Steps in the process are shown in Figure 3.

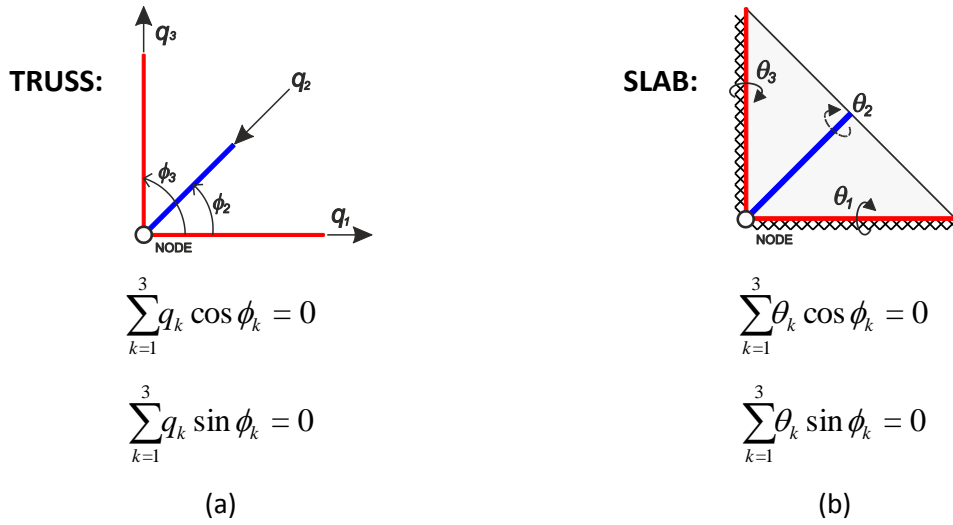


Figure 2: Analogy between (a) truss equilibrium, and (b) slab compatibility at a node

For a problem comprising n nodes and m potential yield-lines, the resulting 'discontinuity layout optimisation' (DLO) formulation can be written as:

$$\text{minimise } \lambda = \sum_{j=1}^m m_p l_j (\theta_j^+ + \theta_j^-) \quad (5a)$$

$$\text{subject to } \left. \begin{array}{l} \sum_{k=1}^{m_i} \theta_k \cos \phi_k = 0 \\ \sum_{k=1}^{m_i} \theta_k \sin \phi_k = 0 \end{array} \right\} \text{ for each node } i = 1, \dots, n \quad (5b)$$

$$\sum_{j=1}^m q a_j \delta_j(\theta_j) = 1 \quad (5c)$$

where Eq. (5b) enforces for each node the geometric compatibility constraint shown in Figure 2 (assuming there are m_i yield-line connections at node i). Also, Eq. (5c) enforces the unit external work constraint (see Gilbert et al.⁸ for further details). This is a linear optimisation problem for which highly efficient solvers exist. This means that problems involving thousands of nodes can be solved in a matter of seconds on a modern desktop PC.

An interesting feature of the DLO procedure is that, at points where potential yield-line discontinuities crossover one another (e.g. see instances of this in Figure 3(c)), compatibility requirements are implicitly enforced. Also, since the location of each potential yield-line is known in advance of the optimisation process, it is possible to locally ascribe m_p values, making it straightforward to model slabs with orthotropic or skew reinforcement.

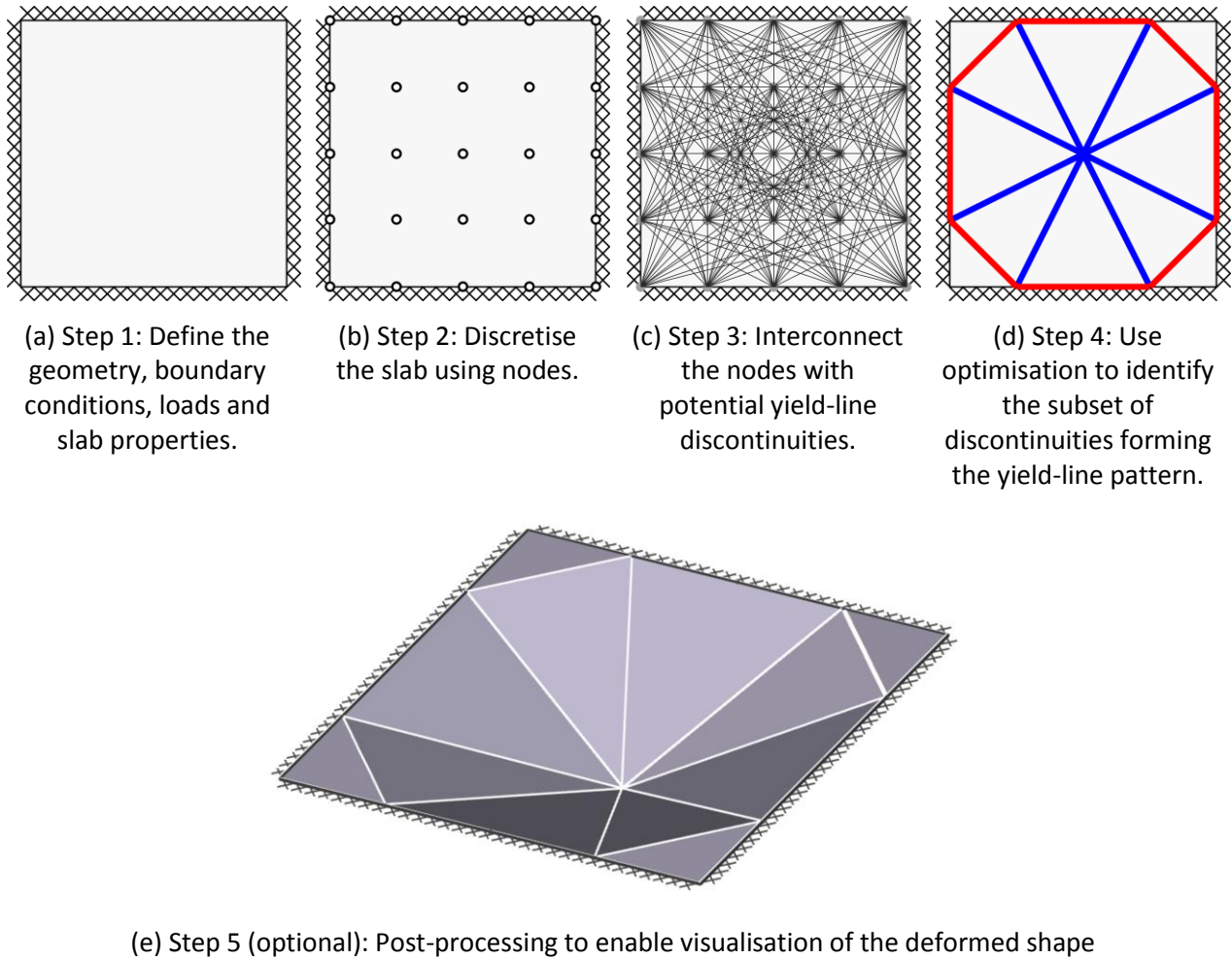


Figure 3: Steps in the automated yield-line analysis procedure

Example 1: Benchmark square slabs

To demonstrate the effectiveness of the automated method it is first applied to various benchmark square slab problems, for which known solutions are available. In each case, the commercially available LimitState:SLAB software⁹ which implements the DLO formulation already outlined was employed. Although some of these problems have been found to be difficult to solve when using rigid finite elements, here solutions well within 1% of the known values were obtained within a few seconds on a modern desktop PC.

Considering first the case of a slab with uniform pressure load and simple supports, Figure 4(a) shows that the familiar 'X' shaped yield-line pattern has been identified as being critical. In this case the exact load factor of $24 (\times m_p / qL^2)$, where L is the side length of the slab) is obtained even when very small numbers of nodes are employed. (Though note that a more complex pattern, incorporating corner fans, is identified when the slab is provided with only bottom reinforcement.)

Figure 4(b) shows the identified yield-line pattern for the uniform pressure load with fixed

supports case (also considered in Figure 3). Here a load factor of 43.052 is obtained, which is quite close to the exact load factor¹⁰ of 42.851. Alternatively, an even closer value can be obtained simply by using more nodes (e.g. a solution of 42.857 was reported by Gilbert et al.⁸). Note that in this case the identified yield-line pattern is somewhat more complex than the patterns typically considered in a hand analysis. This is partly because in a critical yield-line pattern positive and negative yield-lines will be orthogonal to each other, something that, for sake of simplicity, is often ignored in a hand analysis. In addition, the use of a fixed nodal grid in the DLO method means that a single yield-line in the true critical yield-line pattern may be approximately represented by several yield-lines in close proximity to one-another.

Finally, Figure 4(c) shows the identified yield-line pattern for the central point load with fixed supports case; in this case the computed load factor of $12.624 (\times m_p/Q)$, where Q is the magnitude of the point load) is close to the exact load factor² of 4π .

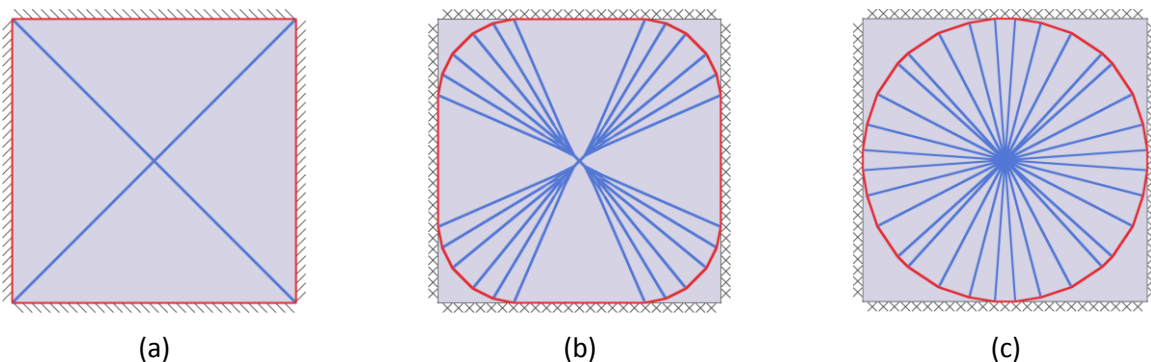


Figure 4: Identified yield-line patterns for uniformly loaded square slabs with (a) simple and (b) fixed supports; (c) pattern for square slab with fixed supports and central point load

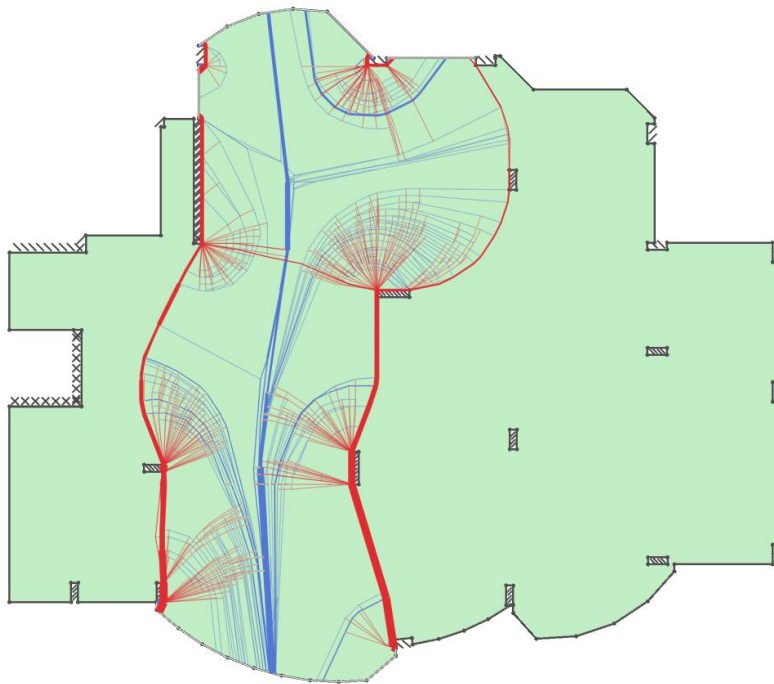
Example 2: Building with irregular floor plate

Hand-based yield-line analysis becomes particularly problematic when complex slab geometries are involved. Kennedy and Goodchild⁴ provide useful advice on the types of mechanism that should be considered, though to account for the increased level of uncertainty involved they recommend that the moment capacity be increased by 15% for the purposes of design, rather than their normal recommended value of 10% (which itself has recently been challenged¹¹).

An example considered by Kennedy and Goodchild⁴ is the relatively complex floor plate of a London apartment block (see Figure 5). The irregular geometry requires that many possible yield-line patterns are considered by hand, which is a time consuming process. Conversely, with the new automated method it is possible to quickly obtain a close approximation of the critical yield-line pattern and associated load factor; see Figure 5(b).



(a)



(b)

Figure 5: Building with irregular floor plate, (a) external view, (b) identified yield-line pattern (obtained assuming simple supports at edges of blade columns)

Example 3: Beam and slab bridge deck

Some years ago Middleton¹² suggested that many highway bridges have a low assessed load carrying capacity not because of inherent weakness, but due to the conservative nature of the elastic methods used to assess them. In a study of 21 local authority bridges initially assessed to

have a capacity of less than 17 tonnes, it was found that over 80% had the capacity to carry at least 38 tonne vehicles when assessed using plastic (yield-line) methods. To facilitate rapid assessment of such bridges a practical software tool, COBRAS, was developed. However, although COBRAS considers a relatively large number of predefined yield-line patterns, there is still a concern that the critical mechanism may be missed. (To address this Jackson and Middleton¹³ recently developed a more general plastic analysis procedure; however, obtaining the critical yield-line pattern necessitated a manual interpretation step.)

Figure 6 shows an example of a four-wheeled vehicle traversing a beam and slab bridge deck, in this case assuming that the wheels act as point loads, and that both beams and slabs behave in a plastic manner. Using the new automated method it is evident that a relatively complex yield-line pattern, of the sort that would very unlikely to be identified by hand, is identified as being critical.

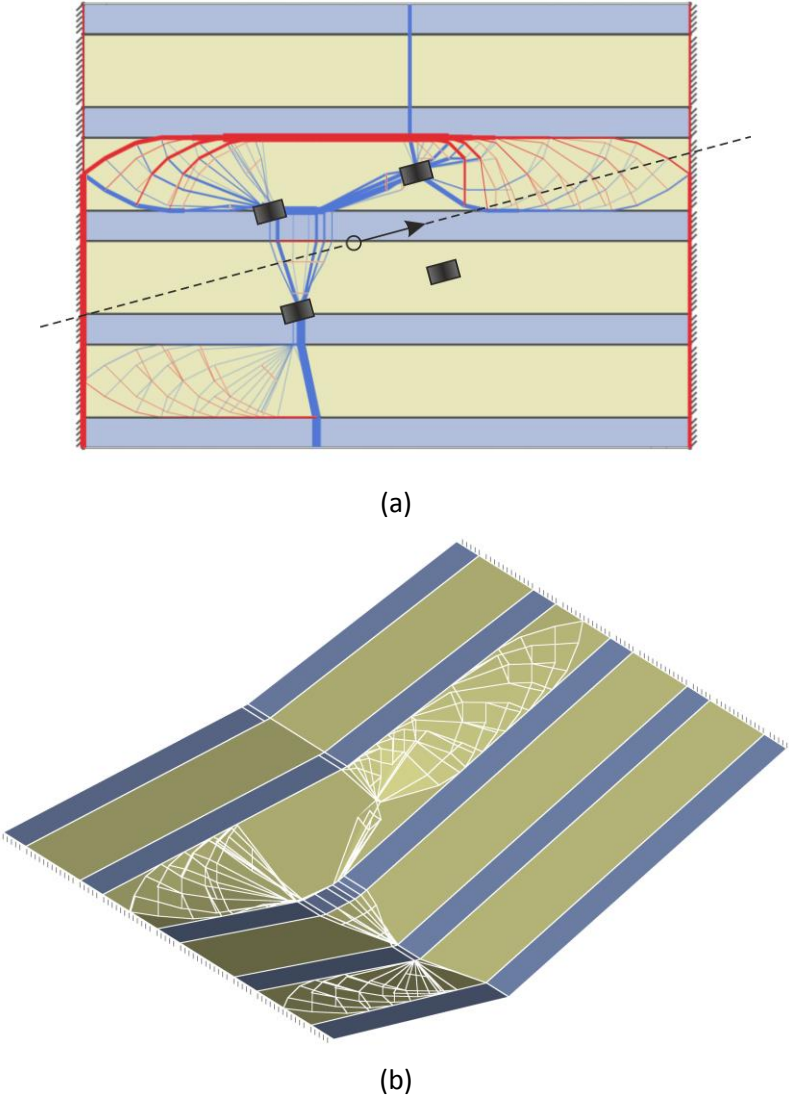


Figure 6: Beam and slab bridge deck, (a) identified yield-line pattern at critical vehicle position; (b) corresponding deformed shape

Conclusions

The yield-line method provides a powerful means of analysing the ultimate (collapse) limit state. Benefits of the yield-line method are that it will often identify additional reserves of strength when applied to the analysis of existing slabs, and to highly economic slabs when used in design. However, the lack of a general computer-based implementation has limited its popularity in recent years. To address this the yield-line method has now been systematically automated, using discontinuity layout optimisation (DLO). This provides structural engineers with a viable alternative to the elastic analysis methods that have become prevalent for collapse analysis in recent years, which can yield excessively conservative results.

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