Comparison of Limit Equilibrium and Limit Analysis for Complex Slopes

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ABSTRACT: Conventional slope stability analysis uses various Limit Equilibrium (LE) methods to determine the minimum factor of safety and its associated critical failure mechanism. These methods frequently assume that collapse will follow a particular assumed geometry, which are effective for simple geotechnical problems, yet may encounter difficulties when considering complex problems. For such complex problems, one effective solution lies in the use of Limit Analysis (LA) based on the upper-bound of plasticity in conjunction with a discretization procedure known as Discontinuity Layout Optimization (DLO), which can be an effective means of determining a critical failure mechanism without the limitation of an assumed slip surface. This paper compares the use of LE (Spencer Method with dynamic programming optimization) and LA for several well-known examples of complex slopes. It is shown LA generally provides slightly lower factors of safety than rigorous LE and manages to handle complexities more effectively without assuming the geometry of the slip surface. It is demonstrated that using the DLO with LA can be applied to various extreme value problems beyond classical slope problems.

INTRODUCTION

Conventional slope stability analysis often consists of various techniques generally employing the Limit Equilibrium (LE) method. This technique utilizes an assumed slip surface and determines its static equilibrium, usually by discretizing the assumed failing soil mass into slices. The vertical and horizontal forces (as well as moment for rigorous methods, such as Spencer's analysis) are then summed for each slice, creating a statically determinate problem following some assumptions. By introducing the notion of the factor of safety for the entire sliding mass, global equilibrium is maintained for a system at the verge of failure. Such methods have proven to be a reliable means for determining the stability of slopes, but are based on pre-chosen slip surfaces of an assumed geometry (i.e. log-spiral, circular, wedge, etc.), critically restricting its use to simple slope and strata geometry. However, algorithms (e.g. Baker, 1980) have been developed to aid analysis of complex geometries and capture critical slip surfaces.

Alternatively, limit analysis (LA) can find agreeable results without the assumption related to statics or of predetermined slip surface geometry, allowing capture of the critical results, even for complex soil profiles and/or geometries. Additionally, use of LA may facilitate the analysis by requiring less iterations in comparison to LE, in turn reducing computational time. Limit Analysis models soil as a material that is perfectly plastic and obeys an associated flow rule (Yu et al., 1999). LA employs a dichotomy of theorems to provide a solution, either upper bound or lower bound plasticity. Lower bound plasticity theorem concerns whether an applied stress does not violate a soil's strength criterion (e.g., Mohr-Coulomb's c, ϕ), stating that in such a case, the soil will not fail or is at the brink of failure (Chen, 2008). Upper bound plasticity, the method employed in this study, states that when the rate of work along an admissible failure surface due to external loads is greater or equal to the work done by internal stresses, the external loading cannot exceed the actual collapse load, hence determining the upper bound of collapse based on plasticity principles (Chen, 2008, Yu et al., 1999). This method is frequently employed iteratively in order to determine the lowest upper bound solution, while this paper employs the Discontinuity Layout Optimization (DLO) Algorithm to establish the critical upper bound collapse mechanism.

It is interesting to note that the equivalency between LA and rigorous LE analysis has been proven analytically to exist by Leshchinsky et al. (1985) when viewed in the context of limit equilibrium. That is, rigorously obtained LA solution also satisfies static force equilibrium for the sliding mass, same as rigorous LE analysis does. However, the converse is not true as LE solution does not necessarily satisfy LA analysis as it does not consider kinematical constraints. It turns out that the LA software used enables one to obtain numerically a free body diagram for each elemental body comprising the failure mechanism essentially showing that the LA rigorous results also satisfy force equilibrium. Hence, the upper bound solution reported in this work also satisfies global equilibrium and therefore, can also be considered as a legitimate rigorous solution in the context of limit equilibrium, as is Spencer's method. Furthermore, the numerical upper bound produces an admissible failure mechanism (in the context of plasticity) which is of no concern in LE.

To further validate the use of Limit Analysis in a practical sense, especially for complex problems, this study compares the critical failure surfaces determined from limit equilibrium analysis reported by Baker (1980), which uses a dynamic programming optimization algorithm, to upper bound LA results utilizing the DLO algorithm. The results found by Baker, which are re-analyzed using LA and compared, range from simple slopes to complex slopes with varying strata, water tables and retaining structures. Baker's reported results are based on a pre-assigned 10 sections along the width of the problem, and evenly spaced nodes along the height of each section. Linear segments between the nodes enable a search which eventually leads to the critical results. It is noted that considering the power of today's computers, the results generated by Baker can practically be based on a more refined discretization. However, Baker's results are considered as critical acceptable benchmark in the context of LE. In this paper, Baker's problems are reanalyzed using the LA method utilizing program LimitState:GEO (Smith and Gilbert, 2007), commercially available software. It employs an algorithm called Discontinuity Layout Optimization (DLO) to determine the critical failure mechanism based on an assignment of evenly spaced nodes to the soil geometry (Smith and Gilbert, 2007). Since slip surfaces pass through the nodes, each possible node-node failure is examined using linear programming optimization to yield the critical failure mechanism. The effectiveness of this tool in comparison to some well-established, accepted LE solutions is discussed furthermore in this paper.

COMPARISON OF LIMIT EQUILIBRIUM TO LIMIT ANALYSIS

Results of LA are compared to those generated by Baker (1980). The methodology used by Baker is detailed in his classical paper and its critical results are considered as benchmark for the problems analyzed. That is, the results are the most critical ones considering the discretization used by Baker in 1980. Hence, his work is ideal when considering the objective of this paper to present outcome comparison of two different approaches.

Comparison 1: Simple, Homogenous Slope

First, a comparison with a simple, relevant example is conducted; i.e., homogeneous, simple slope geometry. Comparing Spencer's stability analysis, run by Baker in 1980, to the LA method utilizing the DLO algorithm (nodal spacing = 0.75 m) provides an initial validation. To compare, the problem in Fig. 1 is used, with the following data:

$\gamma = 20 kN/m^3$, $\phi' = 40°$, $c' = 12\ kPa$, $r_u = 0.5$, H=30 m, $\beta = 27°$ (slope angle)

There is an excellent agreement, both in FS and location of failure surface (Fig. 1).

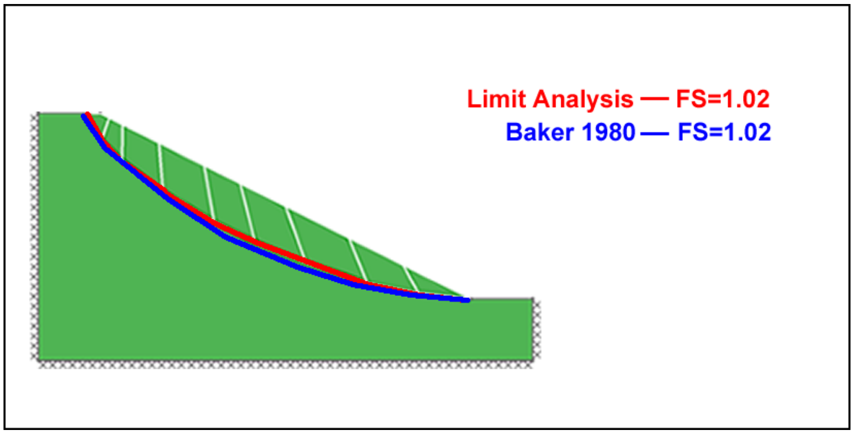

FIG. 1. Comparison 1: LE (Spencer's analysis) and LA.

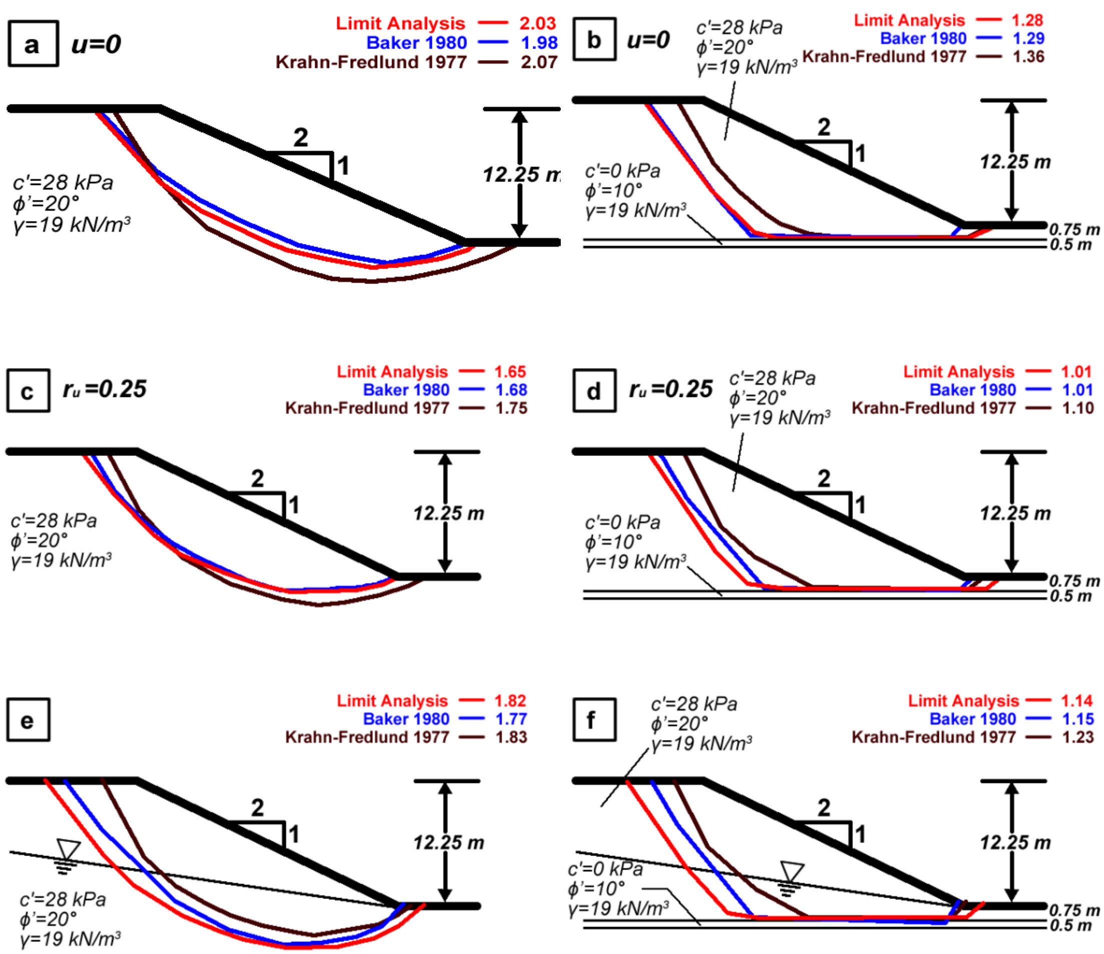

FIG 2. Simple slope analyses a) baseline case; b) underlying weak clay seam with no pore pressure; c) pore pressure, r_u=0.25; d) underlying weak clay seam, r_u=0.25; e) no clay seam, sloped water table; f) clay seam, sloped water table.

Comparison 2: Complex Failure Mechanisms for Simple Geometries

Using simplified geometry, but introducing various factors such as pore pressure (r_u), weak soil seams, or water table add an element of complexity that is relevant yet non-trivial when using slope stability analyses. These more complicated features can start to deviate from the basic assumptions in simplified analyses (assumed circular failure, etc.) as they have clear effects on the failure surface based on their strata. Baker (1980) compares his results with those obtained with the classical analysis presented Kahn and Fredlund (1977), while this discussion compares LA (nodal spacing = 0.5 m) as well. It is noted that Kahn and Fredlund (K-F) classical work also used the Morganstern-Price analysis.

LA finds good agreement, both in FS (error < 3% in all cases) and location of failure surfaces with Baker's prior analysis of a slope under a variety of situations (weak soil seam, water table, etc., Fig.2). Both results tend to find a more critical FS than the results reported by Krahn-Fredlund (1977).

Comparison 3: Complex Soil Geometry

Comparison of various stability analyses for simple but non-trivial, slopes demonstrates acceptable agreement between LA and Spencer's method utilizing a sophisticated search technique for the critical general-shaped slip surface. However, a realistic, yet difficult to define problem includes slopes dealing with complex geometry or layered soils. The layout of the soil profile, as well as the interaction between the various materials can render unconservative results if simplified analyses that assume an a priori failure geometry are utilized. The introduction of a layered soil profile can result in complex collapse mechanisms. In order to demonstrate this concept, an earthen dam with varied soil geometry was used as an example by Baker in 1980, based on actual construction specifications (Fig. 3).

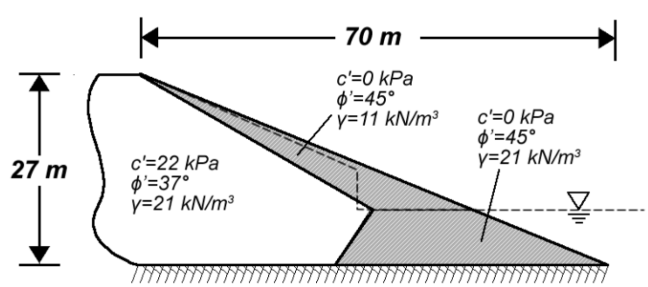

FIG. 3. Original geometry and soil properties of earthen dam.

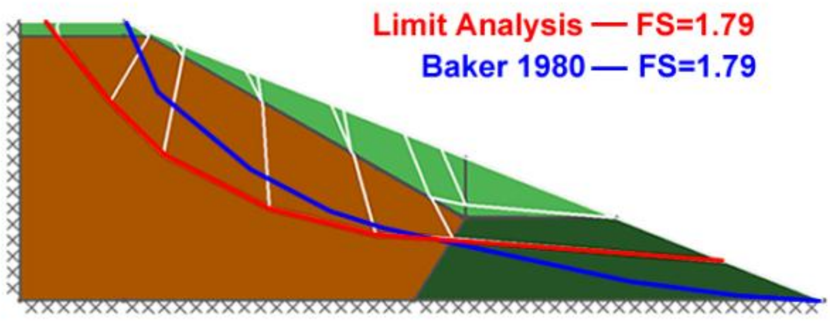

FIG. 4. Comparison of failure surfaces dam example (nodal spacing=0.7 m).

Further analysis using LA (nodal spacing = 0.7 m) finds good agreement in FS (≈1% discrepancy) with Baker's prior results as shown in Fig. 6. The failure surface deviates somewhat as LA shows an exit that is slightly above the toe, unlike Baker's analysis (Fig. 4). It manages to capture a composite failure surface as does Baker's analysis. The deviations of the trace of the slip surfaces can be attributed to coarse discretization used by Baker due to limited computational power in 1980. While FS values are quite close, the slip surfaces are not that close. Clearly, for this problem FS is not very sensitive to the location of the slip surface, agreeing with a well-known observation in slope stability analysis.

Comparison 4: Complex Geometry and Rapid Drawdown

Many realistic geotechnical structures are significantly more challenging to analyze due to their complex geometry (i.e. tiered slopes), varied material properties and structural elements (sand, clay, concrete, steel sheet piles), and groundwater considerations. One example to demonstrate the complexity of this problem is an earth retaining structure with varied backfill, consisting of a concrete retaining wall anchored into an underlying slope with a sheet pile wall, all with a water table that is significantly different on both sides of the retaining wall (ground level on each side, see Fig. 5). While checks of local factors of safety for each retaining structure (sheet pile wall, concrete gravity wall) are needed as part of standard geotechnical design, these values are not representative of global stability. Considering the complicated conditions of this earth structure, a basic global stability analysis utilizing a simplified failure mechanism may provide an unconservative factor of safety. Use of advanced analyses that can capture critical, global general-shaped failures are essential in the analysis of these complex geotechnical problems.

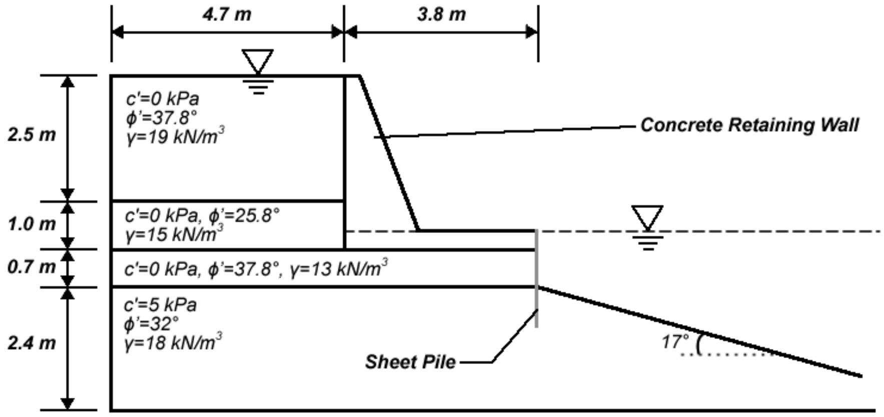

FIG. 5. Geometry of complex geometry with rapid drawdown.

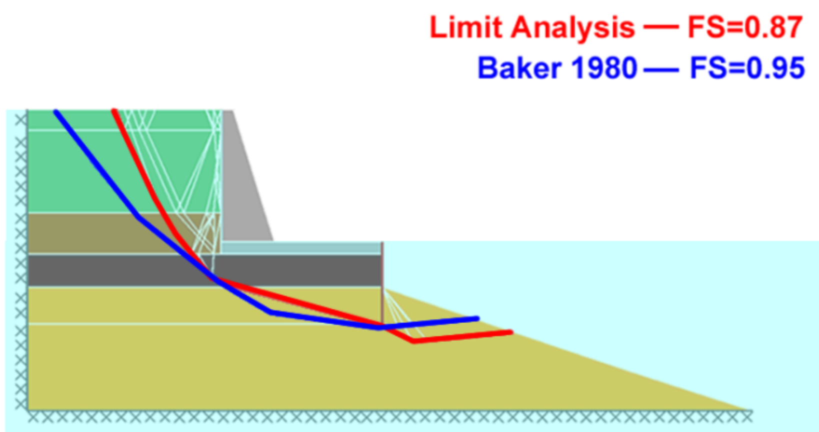

FIG. 6. Failure surfaces for retaining wall/sheet pile placed on slope.

Upon comparison of LA (nodal spacing = 0.7 m) and Spencer's analysis, it is shown that there is generally good agreement (Fig. 6) where the LA method applied to this model yields somewhat smaller FS than the results found by Baker (1980). Using a nodal density of 0.7 m provides a similar failure surface, although there is a discrepancy between its FS and that of Spencer's analysis (8.4%). It is noted that because of computational limitation capacity in 1980, Baker's discretization in using the dynamic programming was rather coarse; it is entirely possible that by using a finer vertical sections, Baker's critical results would have been closer to the LA results.

Comparison 5: Complex geometry combined with free-draining face

The introduction of a slope with varied soil layers, a downstream having a water table at ground surface, a free-draining slope face, and a phreatic surface upstream intercepting the free draining layer at some elevation well above the toe, is a fairly common geotechnical structure, especially for dams. One economical challenge is optimizing the depth of the drainage layer, d, for stability of the slope face, done by Baker (1980). The results are compared to a novel analysis using LA with the DLO algorithm as shown in Fig. 7.

LA encountered some discrepancies between Baker's analysis (3%-15%), often with a lower, more critical FS. The use of a free draining soil at the face of the slope prevents a failure of the weaker backfill, but comes at a cost, necessitating an optimization of thickness of the material while maintaining stability. The analysis still managed to capture similar drainage depth for critical stability (d=0.3 m for Spencer, d=0.4m for LA, Fig. 8), showing that the differences in FS are most likely due to sensitivity of dealing with a purely frictional material combined with water pressure. This is likely the inability to capture a purely surficial failure, a singularity, when no drainage (dry) material is at the face of the slope. This problem could be due to nodal placement and density (nodal spacing=0.75 m); however, this singularity is very difficult to capture using alternative stability analyses (Shukha et al., 2005) as the slope and friction angle coincide. However, LA does manage to show the reduction of the size of the failure mechanism and its change to a near-planar failure with reduced drainage material.

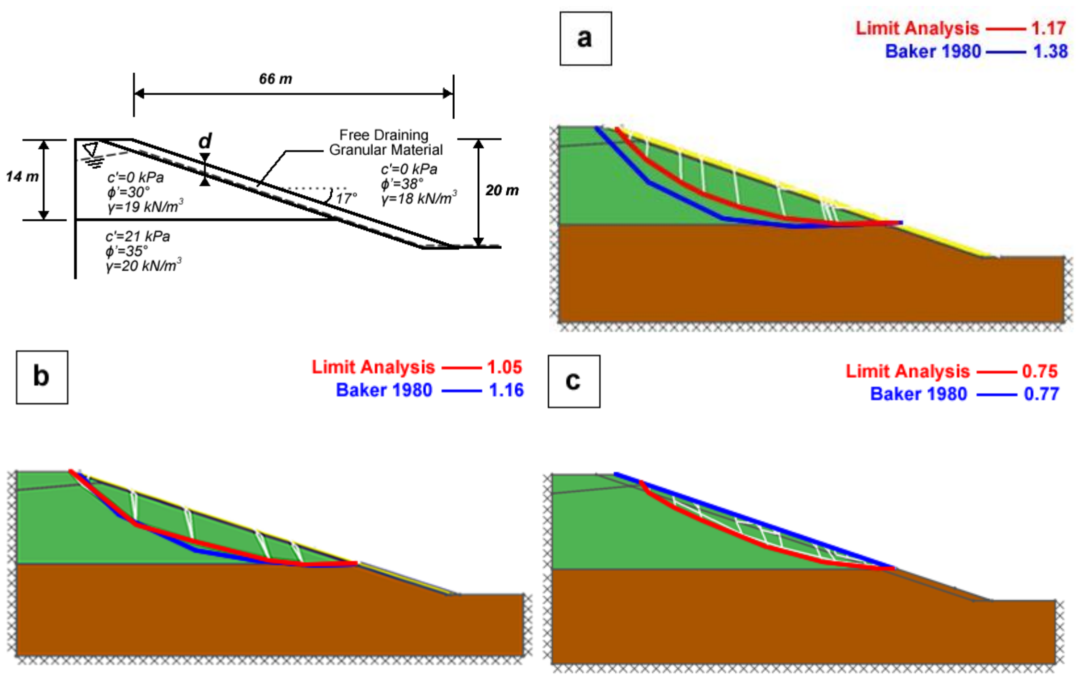

FIG. 7. Top left) Geometry of slope with free-draining face and comparison of failure surfaces when face is a) 1 m; b) 0.5 m; and c) 0 m.

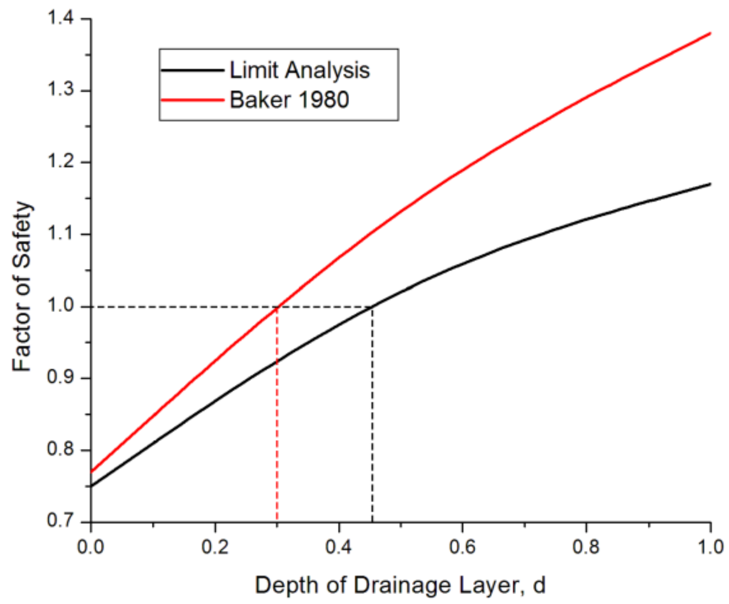

FIG. 8. Graphical comparison of FS from LA and Baker (1980) over varying drainage depth.

CONCLUSIONS

Conventional LE stability analyses like Spencer's method are widely accepted as a mainstream and satisfactory technique of determining the stability of a given slope. However, alternative methods, such as upper-bound plasticity analysis (i.e. Limit Analysis), can provide similar results, especially when employing a discretization technique that allows for the capture of complicated and comprehensive (sliding, overturning, global, composite) collapse mechanisms for non-trivial geotechnical problems. Additionally, employment of Limit Analysis with DLO has potential to be an extremely effective tool in analyzing complex geotechnical structures and their stability. It was shown that Spencer's LE utilizing dynamic programming procedures and Limit Analysis utilizing the DLO algorithm had generally good agreement for the compared examples. The results from LA with DLO were mostly equal to, and in some cases, more critical than those from LE combined with dynamic programming. A comparison of analyses shows that the main advantage of using the LA with DLO is that it does not require one to a priori assume the slip surface shape or location. Conversely, in most LE methods, such an assumption is needed. Allowing the user to determine the stability of an extremely complex geotechnical structure without assuming a simplified, and in turn, possibly unrealistic failure mechanism is advantageous as it provides a more comprehensive analysis requiring less judgment and speculation. It was important to consider the effects of discretization from nodal density when using DLO. It is shown that nodal density can affect discretization of failure surfaces, but generally it is not a major issue. While various methods of LE are still viable means of analyzing stability of slopes, it is shown that LA utilizing DLO is an effective method of determining the stability of complex slopes without an assumed slip surface geometry.

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